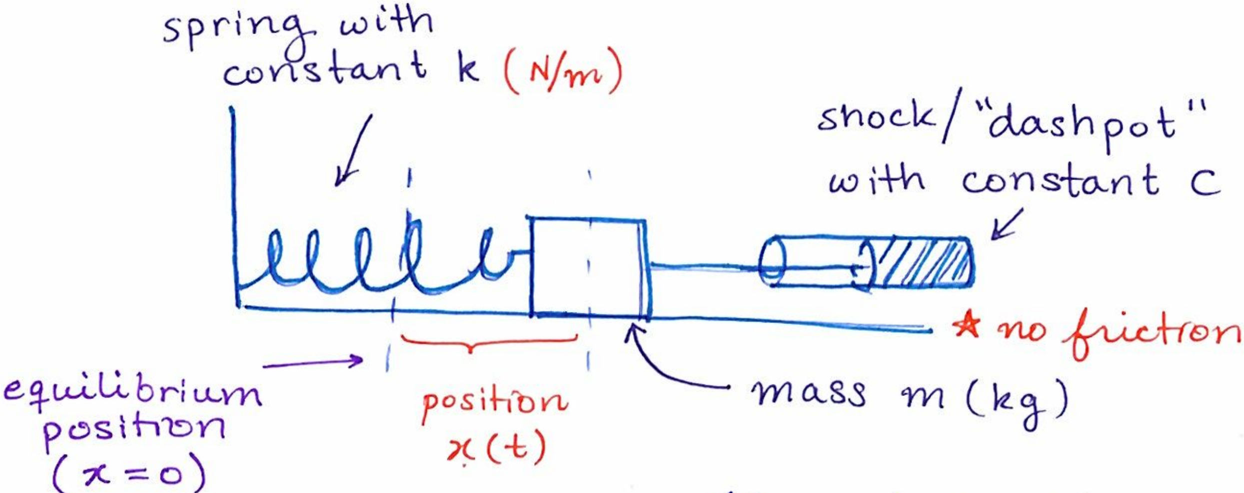
Sec 5.4 Mechanical Vibrations



spring force : $F_s = -\boxed{k}x$ ("Hooke's law")

- k is # Newtons required to stretch spring 1 m

Damping force: $F_d = -cv = -cx'$

(proportional to velocity, like air resistance)

Newton's 2nd law ($F=ma$) $\Rightarrow$ $ma = F_s + F_d$

$\Rightarrow \; mx'' = -kx - cx'$

$\Rightarrow \; \boxed{mx'' + cx' + kx = 0}$

Ex (undamped) Given:

- mass $m = 4\,kg$
- undamped $\Rightarrow$ $c = 0$
- 5 N stretch the spring 25 cm

  $\Rightarrow$ $k = \frac{5N}{0.25m} = 20\, N/m$
- initial conditions $x_0 = x(0) = 0$

  $v_0 = x'(0) = 3$

so eqn is $4x'' + 20x = 0$

$x'' + \boxed{5}x = 0$

$x'' + \boxed{\omega_0^2}x = 0$ $\qquad \omega_0 \overset{DEF}{=} \sqrt{k/m}$

($\omega_0 = \sqrt{5}$ here)

ch. eq. is $r^2 + \omega_0^2 = 0$

$r^2 = -\omega_0^2$

(conjugate root pair) $\quad r = \pm \omega_0 i \quad \to e^{i\omega_0 t} \qquad e^{-i\omega_0 t}$

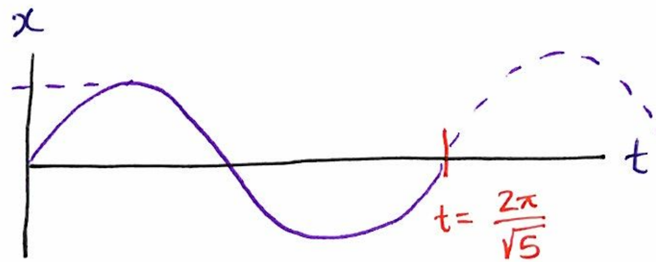
gen sol is $\boxed{x(t) = A \overbrace{\underset{\cos(\sqrt{5}t)}{\cos(\omega_0 t)}}^{Re(e^{i\omega_0 t})} + B \overbrace{\underset{\sin(\sqrt{5}t)}{\sin(\omega_0 t)}}^{Im(e^{i\omega_0 t})}}$

particular solution $x(0) = 0, \quad x'(0) = 3$

... solve for A, B, $\longrightarrow A = 0, \quad B = \frac{3}{\omega_0} = \frac{3}{\sqrt{5}}$

so particular solution is

$$x(t) = \frac{3}{\sqrt{5}} \sin(\sqrt{5}\, t)$$



vocab:

- $\omega_0 = \sqrt{k/m}$ angular frequency ($\sqrt{5}$ rad/s)
- ν ~~or~~ or $f \overset{DEF}{=} \frac{\omega_0}{2\pi}$ frequency $\frac{\sqrt{5}}{2\pi}$ $\left(\frac{1}{sec} = Hz\right)$
- amplitude (max distance/displacement) $3/\sqrt{5}$
- period $T \overset{DEF}{=} \frac{1}{\nu} = \frac{1}{f} = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{5}}$ seconds

Combining to cosine

Ex: same ~~st~~ setup (m, c, k), now $x(0) = 2$
$x'(0) = 3\sqrt{5}$

...

particular solution $x(t) = 2\cos(\sqrt{5}\, t) + 3\sin(\sqrt{5}\, t)$ 1

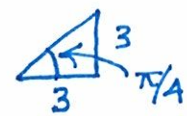
Harder to draw, so we "combine"

Goal: change $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

to $\rightarrow$ $x(t) = C\cos(\omega_0 t - \alpha)$

amplitude C, "phase shift" α

- use $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ in
- algebra ... $\Rightarrow$ $\boxed{C = \sqrt{A^2+B^2}}$, $\boxed{\alpha = \tan^{-1}\left(\frac{B}{A}\right)}$



$$x(t) = 2\cos(\sqrt{5}t) + 3\sin(\sqrt{5}t)$$

$$C = \sqrt{4+9} = \sqrt{13}$$

$$\alpha = \tan^{-1}(3/2) \approx 0.9828 \text{ (rad)}$$

$$x(t) = \sqrt{13}\cos(\sqrt{5}t - 0.9828)$$

General scenario with damping: $mx'' + cx' + kx = 0$

$(c \neq 0)$ ch. eq: $mr^2 + cr + k = 0$

roots (from quad form) $r = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$

$$r = -\frac{c}{2m} \pm \frac{\sqrt{c^2-4km}}{2m}$$

type of root/solution depends on $c^2 - 4km$

(if $c=0$, then $r = 0 \pm \dfrac{\sqrt{-4km}}{2m} = \pm\dfrac{2\sqrt{km}\, i}{2m} = \pm\sqrt{k/m}\, i$ $\pm\omega_0 i$)

roots $r = \frac{-c}{2m} \pm \frac{\sqrt{c^2-4km}}{2m}$

Case 1 : $c^2 - 4km < 0$

Then $\sqrt{c^2-4km} = \sqrt{4km-c^2}\, i$

roots $r = -\underbrace{\boxed{\frac{c}{2m}}}_{\text{"}p\text{"}} \pm \underbrace{\boxed{\frac{\sqrt{4km-c^2}}{2m}}}_{\text{"}\omega_1\text{"}} i = -p \pm \omega_1 i$

(conjugate pair)

In this case, gen solution is (may have to combine to cos)

$$\underline{x(t) = c_1 e^{-pt}\cos(\omega_1 t) + c_2 e^{-pt}\sin(\omega_1 t)}$$

- Oscillations that die-down to zero over time
- This case is called "underdamped"

Case 2 : $c^2 = 4km$

roots $r = -\frac{c}{2m} \pm 0 = -p \pm 0$, $r = -p, -p$

(mult. 2)

gen solution

$$x(t) = c_1 e^{-pt} + c_2 t e^{-pt}$$
$$= \underline{(c_1 + c_2 t)e^{-pt}}$$

- This case is called "critically damped"

- Damper is "just right"
- mass returns to equilibrium in the minimal time

<u>Case 3</u> : $c^2 - 4km > 0$

Here $r = -\frac{c}{2m} \pm \frac{\sqrt{c^2-4km}}{2m}$ makes two

<u>real</u>, <u>distinct</u> roots r_1, r_2 (<0)

gen sol $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

- This is called "<u>overdamped</u>"

Python app...